GENERALIZED ISOTHERMAL COUETTE FLOW OF A NON-NEWTONIAN LIQUID IN A SLOWLY CONVERGING CHANNEL WITH COMPLEX SHEAR

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There is an analysis of the flow of an anomalous-viscosity liquid with a power-law rheological equation in the converging screw channel of a screw pump (an extruder, a mixer, etc.). Circulation of the liquid in the channel is taken into account.

The model usually adopted for the flow of a liquid in the converging screw channel of a screw machine is simple shear flow between two infinite plates inclined at some angle with respect to each other. A pressure gradient acts in the region between the plates. This problem was treated in [1, 2] for a Newtonian liquid, and that for a non-Newtonian liquid, obeying a power-law rheological equation, was treated in [3-5]. Since this model is valid only for screws with a small pitch angle and for liquids whose properties are not greatly different from those of a Newtonian liquid, we feel it is worthwhile to treat the analogous problem for the case of a complex shear flow.

We consider the flow of a liquid with a power-law rheological equation in the slowly converging screw channel of a screw pump. The screw is shown in Fig. 1a. We assume that there are no gaps between the thread of the screw and the housing, and we assume that the initial depth H of the channel is less than its width S and much less than the screw radius. Under these assumptions we can treat the liquid flow within the channel as flow between infinite plates (Fig. 1b). The lower plate is fixed, while the upper plate moves at a velocity V_0. The x axis is along the channel, the y axis is along its width, and the z axis is along its depth. We neglect the inclination of the plates in the y direction. Acting along the x and y axes are pressure gradients $\partial p/\partial x = A_1$ and $\partial p/\partial y = A_2$. We assume that the liquid flow rate Q is given.

A qualitative examination of the flow pattern in a slowly converging channel shows that, by virtue of the continuity condition, the velocity curves and thus the pressure gradients A_1 and A_2 vary along the length of the channel. At certain values of the parameters in the gap between the plates there can exist a cross section $h_x = h_*$, in which we have $A_1 = 0$. In this case, in the region $H \geqslant h_x \geqslant h_*$ the pressure gradient $A_1 \geqslant 0$ reduces the flow rate of the product, while in the region $h \leqslant h_x \leqslant h_*$ the gradient $A_1 \leqslant 0$ increases the flow rate. If A_1 changes sign in the gap between the plates, the pressure initially increases from the entrance toward the exit and then decreases; if, on the other hand, A_1 changes sign beyond the channel $(h_* \leqslant h)$, , the pressure increases continuously from the entrance of the channel to the exit.

We replace the inclined plane by a series of steps consisting of plane regions of length dx parallel to the x axis, each shifted by an amount dz with respect to the adjacent region. Within each such step the decrease in the channel depth leads to the appearance of a pressure drop dp in this region. We assume that within each step the liquid flows as it would between parallel plates. We neglect the velocity component w_z.

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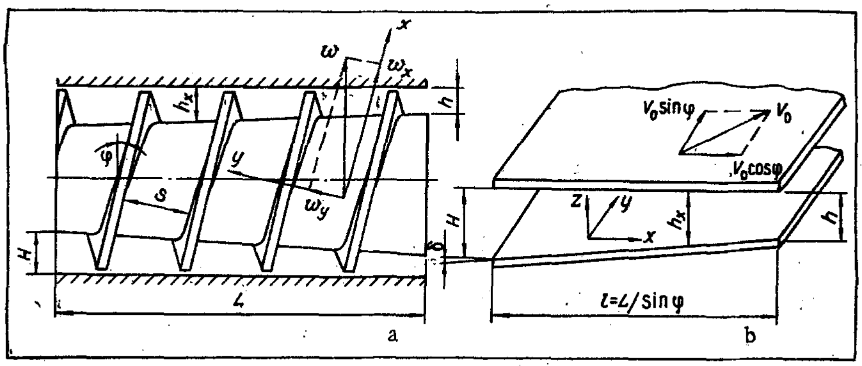

Fig. 1. Screw with a slowly converging channel (a); planar model of this system (b).

As the rheological equation we adopt a power law, which can be written in the following manner for this type of flow [6]:

$$\bar{\bar{\tau}} = B\left(\frac{I_2}{2}\right)^{\frac{n-1}{2}} \cdot \bar{\bar{\Delta}}, \tag{1}$$

where

$$\frac{I_2}{2} = \left(\frac{\partial w_x}{\partial z}\right)^2 + \left(\frac{\partial w_y}{\partial z}\right)^2.$$

The equations of motion along the x and y axes are

$$\frac{\partial \tau_{xz}}{\partial z} = \pm A_1, \frac{\partial \tau_{yz}}{\partial z} = A_2, \tag{2}$$

and their solution is

$$\tau_{xz} = \pm A_1 (z - c_1 h_x),\ \tau_{yz} = A_2 (z - c_2 h_x). \tag{3}$$

Using the condition that the liquid adheres to the plates and the condition that the flow rate in the direction transverse to the channel axis vanishes, we can write the joint solution of Eqs. (1) and (3) in the following form* [7]:

$$v_x = w_x/V_0 = \pm \frac{1}{\alpha}\int_0^{\xi} f(\xi, c_1, c_2, a)(\xi - c_1) d\xi, ^{*)} \tag{4}$$

$$v_y = w_y/V_0 = \frac{a}{\alpha}\int_0^{\xi} f(\xi, c_1, c_2, a)(\xi - c_2) d\xi, \tag{5}$$

*In Eqs. (4)-(6) and below the plus sign corresponds to flow with $A_1 \geqslant 0$, and the minus sign corresponds to flow with $A_1 \leqslant 0$.

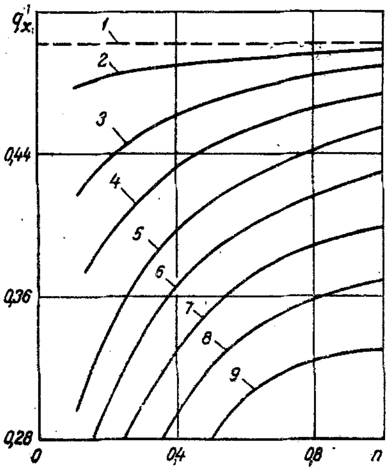

Fig. 2. Dimensionless flow rate q'x as a function of the pitch angle φ and the viscosity anomaly n: 1) φ = 0°; 2) 6°; 3) 12; 4) 18; 5) 24; 6) 30; 7) 36; 8) 42; 9) 48°.

$$\begin{cases} \pm\dfrac{1}{\alpha}\displaystyle\int_0^1 f(\xi,\ c_1,\ c_2,\ a)(\xi-c_1)\,d\xi-\cos\varphi=0, & \text{(a)} \\ \dfrac{a}{\alpha}\displaystyle\int_0^1 f(\xi,\ c_1,\ c_2,\ a)(\xi-c_2)\,d\xi-\sin\varphi=0, & \text{(b)} \\ \pm\dfrac{1}{\alpha}\displaystyle\int_0^1 f(\xi,\ c_1,\ c_2,\ a)(\xi-c_1)(1-\xi)\,d\xi-q_x=0, & \text{(c)} \\ \displaystyle\int_0^1 f(\xi,\ c_1,\ c_2,\ a)(\xi-c_2)(1-\xi)\,d\xi=0. & \text{(d)} \end{cases} \qquad (6)$$

In Eqs. (4)-(6) we have

$$a=A_2/|A_1|,\ \alpha=\frac{V_0}{h_x}\left(\frac{B}{h_x|A_1|}\right)^{\frac{1}{n}},$$

$$q_x=\frac{Q}{V_0h_xS},\ \xi=\frac{z}{h_x},\ f(\xi,\ c_1,\ c_2,\ a)=[(\xi-c_1)^2+a^2(\xi-c_2)^2]^{\frac{1-n}{2n}}.$$

Accordingly, after specifying a definite channel depth h_x, we have a system of four transcendental equations with which we are to determine integration constants c_1 and c_2, the ratio of pressure gradients a, and the unknown α. Knowing the latter, we can determine the longitudinal pressure gradient A_1 in a straightforward manner from

$$\pm A_1=\frac{B}{h_x^{1+n}}\left(\frac{V_0}{\alpha}\right)^n. \qquad (7)$$

The element of power required to overcome the viscous-friction forces under conditions of complex shear over an element of the channel of length dx and width S is given by

$$dN=(\tau_{xz}|_{z=h_x}V_0\cos\varphi+\tau_{yz}|_{z=h_x}V_0\sin\varphi)\,Sdx. \qquad (8)$$

Substituting into Eq. (8) the tangential stresses from (3) at $z = h_x$, taking their sign into account, and using (7), we find

$$dN=B\left(\frac{V_0}{h_x\alpha}\right)^nV_0S\,[a(1-c_2)\sin\varphi\pm(1-c_1)\cos\varphi]\,dx. \qquad (9)$$

We see from system (6) that for the closed-exit regime (q_x = 0) we have $c_1 = c_2 = \xi_m$, in any cross section along the length of the channel, and we have $\alpha = \operatorname{tg}\varphi$. We find an equation for ξ_m after integrating (6d):

$$(1-\xi_m)^{\frac{1+2n}{n}}+\xi_m^{\frac{1+2n}{n}}-\frac{1+2n}{n}\xi_m^{\frac{1+n}{n}}=0. \qquad (10)$$

The joint solution of Eqs. (6a) and (6c) under the condition qx = 0 is

$$\alpha=\frac{A}{b\cos^{b-1}\varphi}\left(b=\frac{1+n}{n},\ A=(1-\xi_m)^b-\xi_m^b\right). \qquad (11)$$

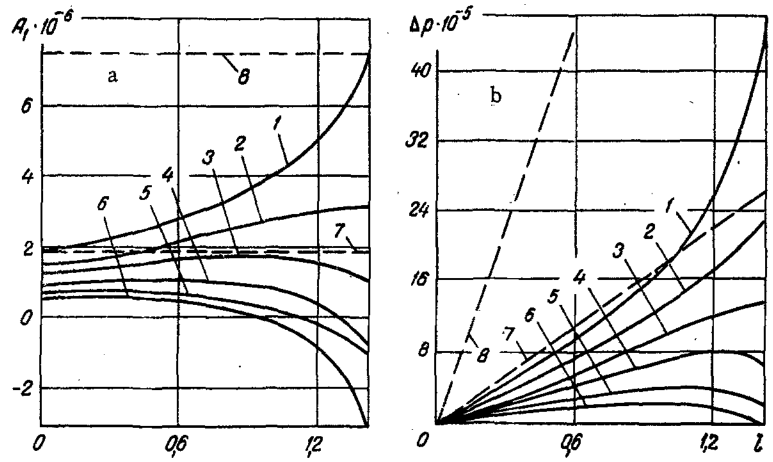

Fig. 3. Profiles of the longitudinal pressure gradient (a) and the pressure drop (b) along the length of the screw channel for various values of the flow rate Q (m^3/sec): 1) Q = 0; 2) $5 \cdot 10^{-5}$; 3) $7.5 \cdot 10^{-5}$; 4) $10 \cdot 10^{-5}$; 5) $12.5 \cdot 10^{-5}$; 6) $Q = Q_{max} = 13.75 \cdot 10^{-5}$: 7) Q = 0 (H = 0.005 m); 8) Q = 0 (H = 0.002 m) A_1, N/m^3; l, m; Δp, N/m^3.

Substituting α from (11) into (7) and (9) successively and integrating the resulting equation over the length of the channel, we find equations for the maximum pressure drop and the power drawn by the screw pump:

$$\Delta p_{\max} = \frac{B\left(\frac{b}{A} V_0\right)^n}{n \cdot \operatorname{tg} \delta} \left(\frac{1}{h^n} - \frac{1}{H^n}\right) \cos \varphi, \tag{12}$$

$$N_{\max} = \frac{B V_0^{1+n} S}{(1-n) \operatorname{tg} \delta} \left(\frac{b}{A}\right)^n (H^{1-n} - h^{1-n})(1 - \xi_m). \tag{13}$$

Integrating the longitudinal pressure gradient A_1 along the length of the channel, we find an equation for the pressure drop between the entrance and the exit of the screw pump:

$$\Delta p = \int_0^l A_1 dx \left(l = \frac{L}{\sin \varphi}\right). \tag{14}$$

Analogously, we find an equation for the total power drawn by the screw:

$$N = \int_0^l \frac{dN}{dx} dx. \tag{15}$$

The depth of the channel at the cross section in which we have $A_1 = 0$ is

$$h_* = \frac{Q}{V_0 S q_x'}, \tag{16}$$

where qx, the dimensionless liquid flow rate, is determined from system (6) by setting $A_1 = 0$.

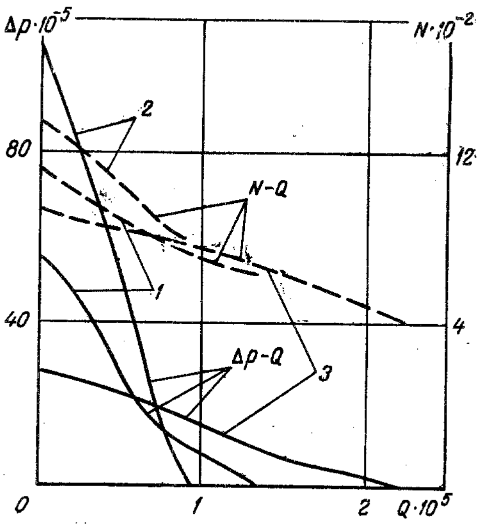

Fig. 4. Pressure drop and power drawn as functions of the flow rate. 1) H = 0.005 m, h = 0.002 m; 2) H = 0.002 m; 3) H = 0.005 m. Here ΔP is in newtons per square meter, Q is in cubic meters per second, and N is in newton-meters per second.

If $h_* \leqslant h$, then we have $A_1 > 0$ over the entire region $H \geqslant h_x \geqslant h$. If, on the other hand, we have $h_* > h$, then over the region $H \geqslant h_x \geqslant h_*$ we have $A_1 \geqslant 0$, and over the region $h \leqslant h_x \leqslant h_*$ we have $A_1 \leqslant 0$.

Figure 2 shows the dimensionless flow rate q'x as a function of the pitch angle φ and the viscosity anomaly n calculated on a computer. We see that as the pitch angle φ increases, the value of q'_x decreases, while it increases with increasing viscosity anomaly —more rapidly, the larger this anomaly. In the case $\varphi = 0$ (the case of simple shear flow) we have $q'_x = 0.5$, independent of the viscosity anomaly. In the case of simple shear Couette flow between parallel plates we have $Q = 0.5V_0hS$, and the velocity profile is a triangle.

To carry out concrete calculations we adopted the following parameter values: H = 0.005 m, h = 0.002 m, $\varphi = 12°$, S = 0.097 m, L = 0.3m (1 = L/sin φ = 1.42 m), B = 200 $N \cdot sec^n/m^2$, n = 0.5, and V_0 = 1 m/sec.

The system of transcendental equations in (6) was solved numerically on a computer; the results are shown in Figs. 3 and 4.

Figure 3a shows the profile of the longitudinal pressure gradient A_1 along the length of the channel for various flow rates Q. We see that in the case Q = 0 (curve 1) the pressure gradient increases continuously from the entrance to the channel to its exit, while in the case $Q = Q_{max}$ (curve 6) it continuously decreases. The point at which the curves intersect the abscissa gives the cross section in which we have $A_1 = 0$. Curves 2-5 are plotted for intermediate flow rates. We see from this figure that as the flow rate increases, the depth of the channel with the vanishing longitudinal pressure gradient shifts from the exit from the channel toward its entrance, reaching its maximum value at $Q = Q_{max}$. In the case of a completely closed exit (Q = 0) we find from Eq. (16) that $h_* = 0$, i.e., that h_* is at the intersection of the moving and fixed planes.

Shown for comparison in this figure, by the dashed curves, are the maximum longitudinal pressure gradients for screws with a constant channel depth (curve 7 corresponds to a screw with H = 0.005 m, while curve 8 corresponds to H = 0.002 m). While the pressure gradient is constant for screws with a constant channel depth, that for screws with a slowly converging channel can vary in different manners, depending on the flow rate.

Figure 3b shows the pressure profile along the length of the screw channel. While for screws with a constant channel depth the pressure increases linearly from the entrance to the exit (curves 7 and 8), the pressure profiles are nonlinear in all cases for screws with a constant channel depth, that for screws with a slowly converging channel. Curve 6 corresponds to a completely open exit ($Q = Q_{max}$); in this case the pressure of the product initially increases to some maximum and then decreases. The pressure drop between the entrance to the channel and the exit from it vanishes in this case, and the maximum on the curve shows the cross section $h_x = h_*$ in which we have $A_1 = 0$.

Figure 4 shows the pressure drop Δp and the power drawn N as a function of the flow rate of the screw pump Q. Curves 1 correspond to a screw with a converging channel, while curves 2 and 3 correspond to screws with a constant channel depth (curve 2 is drawn for H = 0.002 m, while curves 3 are drawn for H = 0.005 m). We see from this figure that the maximum flow rate developed by a screw with a converging channel is less than that for a screw with H = 0.005 m but larger than that for a screw with H = 0.002 m. The same conclusion can be drawn regarding the power drawn. As for the maximum pressure drop, it is smaller for the screw with a variable channel depth than for a screw with H = 0.002 m but larger than that for a screw with H = 0.005 m.

NOTATION

x, y, z cartesian coordinates; H, h_x, h, initial, instantaneous, and final depths of the screw channel; L, screw length; l, S, length and width of screw channel, respectively; φ, pitch angle; V_0, velocity of upper plate; v_x, v_y, dimensionless liquid velocities; w_x, w_y, actual velocities of liquid particles; A_1, A_2, pressure gradients; Q, product flow rate; q_x, dimensionless flow rate; p, pressure; τ, stress tensor deviator; Δ, strain rate tensor; τ_{xz}, τ_{yz}, components of stress tensor; I_2, quadratic invariant of strain rate tensor; B, n, rheological constants; N, power; δ, angle between the lower plate and the upper plate.

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